GRADED DIFFERENTIAL GEOMETRY WITH APPLICATIONS

The concept of *graded supergeometry* originates from the idea of supersymmetry which was born in physics. Apparently, a unified theory of strong, weak, electromagnetic, and gravitational interactions can be constructed in the language of supermanifolds. The *supercalculus* is designed for work with variables endowed with *parity*, which can be even (degree 0) or odd (degree 1). The super-analogs of the spaces $\mathbb{R}^n$ with coordinates $(x^1, \ldots, x^n)$ are *superspaces* $\mathbb{R}^{p|q}$ with *supercoordinates* $(x^1, \ldots, x^p, \xi^1, \ldots, \xi^q)$, where x^i are even and ξ^a are odd. Even coordinates commute with all coordinates, $x^i \cdot x^j = x^j \cdot x^j$ (like for standard coordinates) and $x^i \cdot \xi^a = \xi^a \cdot x^i$, while odd coordinates anti-commute among themselves, $\xi^a \cdot \xi^b = -\xi^b \cdot \xi^a$. These properties of supercoordinates we call *sign rules*. The need for calculus with anti-commuting variables became clear when physicists discovered that every elementary particle is either a *fermion* (e.g. electron) or a *boson* (e.g. the light particle, photon), and the difference lies in the collective behavior; bosonic fields commute and fermionic fields anti-commute. This implies that two fermions cannot be in the same physical state.

The fundamental observation made by F.A. Berezin in 60's is that one can develop a differential calculus and differential geometry in such "super" setting. Locally, in supergeometry and supercalculus one works with differentiable functions (this has an appropriate meaning) depending on supercoordinates. Accordingly, partial derivatives and differential operators in supercoordinates are also defined with a use of signs rules, taking into account their parities. Note that, due to supercommutativity, differentiable functions can only depend polynomially on odd coordinates. Global superobjects (supermanifolds) are constructed by analogy with ordinary manifolds by gluing together local supercoordinate charts by means of some invertible differentiable mappings (transition functions). This idea of *supermathematics* is widely used in various fields of science, especially in physics, including quantum mechanics and quantum field theory.

If a privileged family of supercoordinates is endowed, in addition to parity, with *weights* (which are usually integers), we speak of a local *grading*. However, while in the supercalculus the (super)algebra of differentiable functions is uniquely determined by the conditions of differentiability and supercommutativity, in the case of $\mathbb{Z}$-gradings there is an ambiguity in the choice of admissible functions. For non-negative or non-positive gradings, it was customary to consider functions that depend polynomially on coordinates of non-zero weight and smoothly on the rest of coordinates. For a $\mathbb{Z}$-grading, the polynomial functions turn out to be insufficient. In this case, there are two working approaches: the first one allows for graded formal series in variables of non-zero weight, while the second approach permits any smooth functions, but it is required that the differentiable mappings preserve the *weight vector field* ∇. The latter is defined in homogeneous graded coordinates x^i with weights w_i by $\nabla = \sum_i w_i \, x^i \, \frac{\partial}{\partial x^i}$ and contains full information about the grading. It can be shown that the two types of graded manifolds are related. Even in the pure even case, graded manifolds have found applications in various fields, for example, in mechanics and in the theory of exterior differential systems.

It is not strange that, in contrast to the even case, in the "superworld" some linear operators are not self-commuting in the sense that the super commutator $[A, A] = AA + AA = 2A^2$ may not vanish for odd A. If an odd derivation Q of weight 1 of the algebra of functions on a graded supermanifold satisfies the condition $[Q, Q] = 2Q^2 = 0$, then we call Q a *homological vector field* (the name has its roots in homological algebra), and the graded supermanifold a *dg-manifold*. A well known example is the de Rham derivative on the superalgebra $\Omega(M)$ of differential forms on a (standard) manifold M, which can be viewed as the algebra of smooth superfunctions on a "super" version of the tangent bundle TM. The idea of dg-manifolds comes from theoretical physics, namely from the so called *BV-BRST formalism*, which is a powerful tool for studying physical systems with a degenerate Lagrangian. Today, this concept has found wide applications in mathematics and physics, potentially describing the physical laws that underlie the Universe.

The project is devoted to studies on graded supermanifolds and their relations, various geometrical structures on them, and possible applications. The reason is that there is still a lot of open questions in the subject, important from the point of view of analysis, geometry, and many areas of physics. We hope to solve a big part of them and publish the results in highly recognized international journals.